SINGULARITIES OF FOLIATIONS AND GOOD MODULI SPACES OF ALGEBRAIC STACKS

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We work over an algebraically closed field $\Bbbk$ of characteristic zero.

There is a striking similarity between the Minimal Model Program and the construction of quotient spaces. Let G be a group acting on a variety X. It is not always possible to construct the quotient space, however, it is always clear what the field of fractions of such space should be: the G-invariant rational functions on X. Thus, constructing a quotient space can be interpreted as choosing the most appropriate representative in a fixed birational equivalence class. More generally, we may ask the following vague

Question. *How are the Minimal Model Program and its various singularity types related to the construction of quotient spaces?*

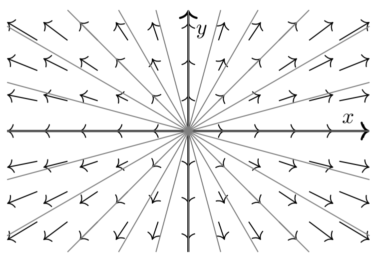
This can be approached in several ways (see for instance [Tha96] and [BGLM24]). As far as we are concerned, we want to understand the following phenomenon on $X = \mathbb{A}^2$.

Let $\mathbb{G}_m$ act on X with weights $(1,1)$. Let $\mathbb{G}_m$ act on X with weights $(1,-1)$.

The induced foliation $\mathscr{F}$ at a point $x_0 \in X$ is the tangent space of the orbit of x_0.

One can see that $\mathscr{F}$ is globally generated by the vector field

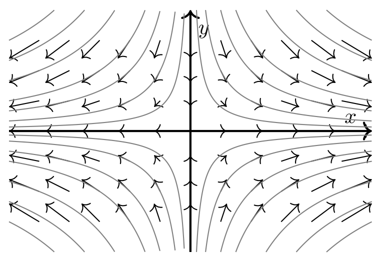
$$\partial := x\partial_x + y\partial_y.$$



The origin is a log-canonical singularity of $\mathscr{F}$ and all other points are regular.
The generic orbit is not closed.
$\mathbb{G}_m$ acts on $U := X \smallsetminus \{0\}$ and the quotient stack $[U/\mathbb{G}_m]$ is $\mathbb{P}^1$.
We obtain a rational map

$$f : X \supseteq U \to \mathbb{P}^1$$
$$(x,y) \dashrightarrow (x:y).$$

$$\partial := x\partial_x - y\partial_y.$$



The origin is a canonical singularity of $\mathscr{F}$ and all other points are regular.
The generic orbit is closed.
$\mathbb{G}_m$ acts on $U := X \smallsetminus \{xy = 0\}$ and the quotient stack $[U/\mathbb{G}_m]$ is $\mathbb{A}^1 \smallsetminus \{0\}$.
We obtain a rational map

$$f : X \supseteq U \to \mathbb{A}^1 \smallsetminus \{0\} \subseteq \mathbb{P}^1$$
$$(x,y) \dashrightarrow (x : 1/y).$$

We can resolve the indeterminacy locus of f by blowing up the origin $\pi : Z \to X$.
We get a morphism $g : (Z, \Delta) \to \mathbb{P}^1$, where $\Delta \subseteq Z$ is the exceptional divisor of π.
Since the origin is a fixed point, the $\mathbb{G}_m$-action lifts to Z and g is $\mathbb{G}_m$-invariant.

Note that $g(\Delta) = \mathbb{P}^1$.
This shows that g cannot descend to a $\mathbb{G}_m$-invariant morphism $f : X \to \mathbb{P}^1$.
The good moduli space of the stack $\mathcal{X} := [X/\mathbb{G}_m]$ is $\operatorname{Spec} \Bbbk$, however $\mathcal{X}$ does not contain any stable point.

Note that $g(\Delta)$ is a point.
This shows that g descends to a $\mathbb{G}_m$-invariant morphism $f : X \to \mathbb{P}^1$.
The good moduli space of the stack $\mathcal{X} := [X/\mathbb{G}_m]$ is $\mathbb{A}^1$, and $\mathcal{X}$ contains an open dense subset of stable points.

In general, we have the following.

Fact ([MP13]). *Let X be a normal scheme and let $x \in X$ be a point. Suppose that ∂ is a vector field on X such that x is ∂-invariant. Then the foliation generated by ∂ is log-canonical if and only if the linear part of ∂ is non-nilpotent.*

The linear part of a vector field around an invariant point is intrinsically defined. In our examples, the linear part of the vector fields are given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

respectively, hence they are both log-canonical.

We may wonder if we can distinguish canonical and log-canonical singularities.

Conjecture ([CS25]). *Let X be a projective variety and let $\mathscr{F}$ be an algebraically integrable foliation on X with canonical singularities. Then there exists a projective morphism $f : X \to Y$ to a projective variety Y such that $\mathscr{F} = \mathscr{T}_f$.*

Recall that a foliation is algebraically integrable if the generic leaf/orbit is algebraic. In our examples, the quotient spaces are $\operatorname{Spec} \Bbbk$ and $\mathbb{A}^1$ respectively.

1. Main Results

Let $\mathcal{X}$ be an algebraic stack with affine stabilisers and finite generic stabiliser.

Main Theorem. *Let $x \in \mathcal{X}(\Bbbk)$ be a point. Then,*

(1) if $\mathcal{X}$ has relative log-canonical singularities at x, the stabiliser group of x is an algebraic torus and a good moduli space exists étale locally on $\mathcal{X}$, and
(2) if $\mathcal{X}$ has relative canonical singularities at x, the locus of stable points with respect to the local good moduli space is non-empty.

Recall that, a local coarse moduli space exists when the stabiliser is finite ([KM97]) and a good moduli space exists when the stabiliser is reductive ([AHR20]). We obtain the following table.

Type of Singularity	**Stabiliser Group**	**Local Moduli Space**
Regular	Finite	Coarse
Canonical	Toric	Stable
Log-Canonical	Toric	Good

We deduce two corollaries.

Corollary A. *Let G be an affine algebraic group acting with finite generic stabilisers on a scheme X. Suppose that the foliation induced by G is log-canonical at x. Then the stabiliser group of x is an algebraic torus.*

Corollary B. *Let $X \to \mathcal{X}$ be a smooth presentation. Suppose that the induced foliation $\mathscr{F} := \mathscr{T}_{X/\mathcal{X}}$ is canonical at x. Then there exists an étale neighbourhood $U \to X$ of x and a morphism $U \to Q$, such that $\mathscr{F}|_U = \mathscr{T}_{U/Q}$.*

We can now build on the results of [MP13]: if we then assume that $\mathscr{F}$ is induced by a smooth presentation of an algebraic stack, such as in Corollary A, then part (1) of our Main Theorem implies that $\mathscr{F}$ is locally induced by a torus action, thus, in particular, it is non-nilpotent.

Furthermore, Corollary B states that, when $\mathscr{F}$ is induced by a smooth presentation of an algebraic stack, the Cascini–Spicer conjecture is true locally on X.

2. Main Ideas

We first need to define foliations and relative singularities.

Definition. A *foliation* on X is a coherent subsheaf $\mathscr{F}$ of the tangent sheaf $\mathscr{T}_X$ which is involutive, i.e. closed under the Lie bracket.

We think of foliations as infinitesimal stacks ([Bon25]). Note that a foliation is not necessarily saturated.

Definition. $\mathcal{X}$ is *relatively (log-)canonical* if there exists a smooth presentation $X \to \mathcal{X}$ such that the induced foliation $\mathscr{F} := \mathscr{T}_{X/\mathcal{X}}$ is (log-)canonical. Since (log-)canonical singularities are insensitive to smooth morphisms, this is independent of the presentation chosen.

The main ingredient is semistable reduction. Variants of this tool, such as [AK00, Theorem 0.3], have already been successfully employed in the study of the Minimal Model Program for algebraically integrable foliations in [ACSS22]. We use *functorial semistable reduction* ([ATW20, Theorem 1.2.17]) to find a diagram

$$\begin{array}{ccc} Z & \xrightarrow{g} & B \\ {\scriptstyle\pi}\downarrow & & \\ X, & & \end{array}$$

where

(1) π is an alteration,
(2) g is semistable, and
(3) $\pi^*\mathscr{F}$ is naturally a foliation on Z contained in the log-tangent sheaf $\mathscr{T}_g$.

We show that, if $\mathscr{F}$ is log-canonical, $\pi^*\mathscr{F} = \mathscr{T}_g$, else we would extract a divisor with negative log-discrepancy.

But now, since g is semistable, $\mathscr{T}_g$ is locally given by a faithful torus action. This allows us to conclude that the same is true for $\mathscr{F}$.

Remark. There are plenty of technical difficulties:

(i) We want g to be semistable, as opposed to merely logarithmically smooth. This ensures that the torus action is faithful. The price to pay is that π is an alteration, as opposed to a modification.
(ii) We need to keep track of the locus where $\mathscr{F}$ is not regular. We do so by adding a boundary divisor and using logarithmic geometry.

(iii) The functorial semistable reduction only produces a Deligne–Mumford stack Z. For this reason, we must work with foliations in this generality.
(iv) In order to deduce that $\mathscr{F}$ is locally given by a torus action, we must use a simple version of Borel's fixed point theorem on Deligne–Mumford stacks.

3. Further Questions

We start with simpler questions and then proceed to more interesting questions.

Question 1. Is the converse of part (1) of our Main Theorem true? If $\mathscr{F}$ is induced by a faithful torus action, is $\mathscr{F}$ log-canonical?

Question 2. Is the converse of part (2) of our Main Theorem true? If $\mathscr{F}$ is log-canonical and is induced by a morphism, is $\mathscr{F}$ canonical?

Question 3. Can we extend our Main Theorem to treat stacks with non-finite generic stabilisers? This involves working with Lie algebroids rather than foliations.

Question 4. Can we extend our Main Theorem to treat algebraically integrable foliations? This would give a partial generalisation of [MP13] in higher rank.

Question 5. Can we extend our Main Theorem to treat arbitrary foliations in a formal neighbourhood? We would need a formal semistable reduction result and work with *formal good moduli spaces.*

Question 6. When $\mathscr{F}$ is canonical, can we glue the local quotients in our Main Theorem, in order to obtain an algebraic space? We would probably need a global assumption on $(X, \mathscr{F})$, such as requiring $K_{\mathscr{F}}$ to be nef.

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